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On the Tables of Single and Annual Assurance Premiums published by the late Mr. William Orchard, and on a Theoretical Table of Mortality proposed by him. By PETER GRAY, Esq., F.R.A.S.

[Read before the Institute, 28th April, 1856, and ordered by the Council to be printed.]

THE values which it has been heretofore customary to tabulate for actuarial purposes, in connection with specified rates of mortality and interest, are those of annuities on single and joint lives at all ages ; and from these, by operations more or less complex, and with the aid of subsidiary tables, the actuary has had to form such other values as might be required in the solution of any particular problem in hand. Of the values thus requiring to be formed, those of the single and annual premiums for assurance occur perhaps more frequently than any others ; and hence the desirableness of simplifying as much as possible the operations by which these are deduced from the corresponding annuities. To effect this simplification was the object of Mr. Orchard's work, in which, as will be admitted by all competent to form an opinion on the subject, the author has been completely successful. The arithmetical operation heretofore necessary has been entirely superseded, and the required values are found simply by inspection.

Let a denote the present value of an annuity of one pound on any status, and A the present value or single premium of an assurance of one pound on the same status ; then are these quantities connected by the following equation :—

$$A = v - (1-v)a \quad . \quad . \quad . \quad (1)$$

Now this is an equation of the first degree; and from this it follows that the values of A corresponding to equidistant values of a will also be equidistant, and consequently will be formed by the simple continued addition to or subtraction from an initial value of a constant quantity. Thus, let a become $a + \Delta a$, and let the corresponding value of A be $A + \Delta A$: then, by (1),

$$A + \Delta A = v - (1-v)(a + \Delta a) \quad . \quad . \quad . \quad (2)$$

Hence, by subtraction,

$$\Delta A = -(1-v)\Delta a; \quad . \quad . \quad . \quad (3)$$

and this is the constant difference of the series of values of A , corresponding to a series of values of a whose constant difference is Δa .

In the tables Δa is taken equal to $-\cdot 01$, and ΔA therefore is equal to $-\cdot 01(1-v)$; by the continual subtraction of which quantity, taken positively, from the assurance value corresponding to annuity 0, which, according to (1), is v , the tables for the different rates of interest are respectively formed.

Again: if π denote the annual premium for an assurance of one pound on a status, the present value of an annuity of one pound on which is a , then

$$\pi = \frac{1}{1+a} - (1-v) \quad . \quad . \quad . \quad (4)$$

The variable a here occurring in a denominator, the difference of this expression is of a more complex form than that of the expression for the single premium, and the table does not admit of formation by the continual addition or subtraction of a *constant* quantity. The formation, nevertheless, can be very easily effected.

Thus, when a becomes $a + \Delta a$, π becomes $\pi + \Delta \pi$; and we have by (4),

$$\pi + \Delta \pi = \frac{1}{1+a+\Delta a} - (1-v);$$

whence, by subtraction,

$$\Delta \pi = \frac{1}{1+a+\Delta a} - \frac{1}{1+a};$$

from which it appears that the quantities by the continual addition of which to v (the value of π when $a=0$) the successive values of π will be formed, are the differences of the series of reciprocals belonging to the successive values of $1+a$. And these differences can be readily obtained from Barlow's Table of Reciprocals.

In regard to this series of differences, too, it deserves to be noted that they are independent of the rate of interest. When

formed for use with one rate, therefore, they can be immediately applied to any other.

By the skilful application of the principles here exposed, the tables were formed. In point of fact, so well devised were the arrangements for the work, that, as is consistent with the knowledge of the present writer, the construction was effected in the form the tables now occupy, and the proofs were set up from the original computations.

The argument series in both sets of tables consists of values of a , increasing from 0 upwards by a constant difference of .01. The premiums corresponding to these values only are, therefore, strictly speaking, to be found by inspection. Tables of proportional parts, however, being added, the premiums corresponding to intermediate annuity values are very easily formed.

Annuity values are frequently employed to three decimal places only. It may be well to point out, however, that the fourth decimal place, especially where the higher rates of interest are concerned, exercises a sensible influence on the value of the single premiums. Thus, at 6 per cent., the tabular difference corresponding to .01 being .057, that corresponding to .0005 (the limit of the error in the annuity value when the fourth decimal place is rejected) will be $\frac{.057 \times .0005}{.01} = .00285$. It thus appears

that, when the rate of interest is 6 per cent.; the rejection of the fourth decimal place in the annuity value will give rise to an error in that of the single premium for assurance, which may equal but cannot exceed .003. Hence, when the greatest attainable accuracy is desired, it will be well to make use of the fourth decimal place, by a second entry with it in the table of proportional parts.

Another point in the use of the tables seems to deserve notice. I believe it is a fact, that few experienced arithmeticians will feel the same confidence in the *unexamined* result of a subtraction as in that of an addition of two lines, even although the less of the two numbers operated upon consist of so few as two figures.* It will therefore be well generally to use addition in interpolating for values of a intermediate to those tabulated. This is done by entering the tables with the value of a next *higher* than that given, and *adding* to the result the p.p. corresponding to the complement

* I had written here, "few *even* of the most experienced arithmeticians;" but a very little consideration has induced me to alter the expression to its present form. I believe that one result of experience in arithmetical operations is to beget caution, since it reveals the existence of liabilities to error not previously suspected.

to 10 of the third decimal place, or the complement to 100 of the third and fourth decimal places, according as three or four decimal places are used.

These remarks will be illustrated by the following examples:—

I. The value of an annuity on a certain status, at 5 per cent., is 7·632; required the single premium for an assurance of £100 on the same status.

$$\begin{array}{r} \text{Single premium for } 7\cdot64 = 58\cdot857 \\ \text{p.p. for } 8 = 38 \\ \hline 58\cdot895 \end{array}$$

II. Required the same for annuity value 7·6324.

$$\begin{array}{r} \text{Single premium for } 7\cdot64 = 58\cdot857 \\ \text{p.p. for } 76 = 36 \\ \hline 58\cdot893 \end{array}$$

III. Required the annual premium for annuity 7·6324, 5 per cent.

$$\begin{array}{r} \text{Annual premium for } 7\cdot64 = 6\cdot812 \\ \text{p.p. for } 76 = 10 = 1\cdot3 \times 7\cdot6 \\ \hline 6\cdot822 \end{array}$$

Here the same result would have been obtained by using the first three decimal places only.

Obvious as the utility of these tables now appears, and desirable as the possession of the power they confer must always have been felt to be, the idea of constructing them does not seem to have presented itself to any one before Mr. Orchard;* and the construction could hardly have fallen into better hands. Mr. Orchard was a most expert arithmetician and skilful computer. His arrangements for the construction of these tables, in which he was entirely unassisted, were admirably devised, and thoroughly and systematically carried out; and, so far as I am aware, only one error has been discovered in the tables.†

* Singularly enough, at the meeting of the Institute of Actuaries next succeeding that at which Mr. Orchard presented a description and specimens of his tables, there was exhibited in print a copy of "Conversion Tables, by William Wood, F.I.A., Secretary to the Scottish Amicable Life Assurance Company." The object of Mr. Wood's tables was precisely the same as that of Mr. Orchard's. The idea on which they are founded also is the same; but it is carried out by Mr. Wood in a different way, and is applied by him to fewer rates of interest, while his results are presented in certainly a much less convenient form. See p. viii.

† Page 47, 3 per cent. The single premium corresponding to 15·16 should be 52·932.

Mr. Orchard was cut off at the early age of thirty. From the age of fifteen he held an appointment in the counting-house of a Manchester warehouseman in the city of London; and during the last ten years of his life, till within a few months of his death, he occupied the responsible position of cashier and book-keeper.

Mr. Orchard was a mathematician of no mean order. He was entirely self taught. Endowed by nature with an uncommon aptitude for mathematical investigation, he early entered on the study, and was eminently successful. During several years before his death he had acquired a very considerable command of the resources of the higher analysis, and there were few of the inquiries in which it has been employed of which he had not some knowledge. He had entered upon several original investigations, which, had he lived to complete them, and to give their results to the world, would have gained for him an honourable place in the list of English non-academical mathematicians.

Amongst subjects that at various periods occupied a good deal of Mr. Orchard's attention may be mentioned Continued Fractions, Arbogast's Method of Derivation, the Solution of Equations of all orders by Logarithms, and the Theoretical Representation of the Law of Mortality. In regard to the first three of these subjects, which were left in a very undeveloped state, it is not necessary here to say more than that Mr. Orchard believed he had made some improvements in continued fractions and in Arbogast's method, and that in his method for the solution of equations he was not anticipated by any previous writer. The subject last mentioned, viz., the Theoretical Representation of the Law of Mortality, is that which most recently occupied Mr. Orchard's attention, and his papers relating to it were fortunately left in a rather more forward state than those relating to the other subjects named. It is proposed here to give a brief account of his scheme, so far as it can be gathered from his papers.

The desirableness of being able to express the curve of mortality by means of an equation between two variables is sufficiently testified by the fact of many eminent mathematicians having given their attention to the subject. Among these may be mentioned Lambert, Duvillard,* Thomas Young,† Babbage,‡ Gom-

* Lacroix, *Traité Élémentaire du Calcul des Probabilités*, pp. 199, 200.

† Phil. Trans. 1826. (See also the recently published *Life of Dr. Young*, by Dr. Peacock, chap. xiii.)

‡ I copy the following memorandum made by myself, under date May 3rd, 1845:—

“The following formula I copied from an original letter of Mr. Babbage to Mr. Baily

pertz,* Edmonds,† Farr,‡ &c. Some of the writers named, by the aid of philosophical considerations of a high degree of probability, have deduced formulæ which can be made to represent very closely the indications of our best tables of mortality; but, generally speaking, the formulæ deduced have been such as were but ill adapted to facilitate the computation of the value of life contingencies. It is to this last point that Mr. Orchard chiefly directed his attention. Leaving, in a great measure, out of view theories as to the law in accordance with which the variation in the intensity of mortality is regulated, he sought to find a *simple algebraical* relation which should passably well represent some of our best tables. The table formed in accordance with the relation finally adopted will be found at pp. 268, 269; and I proceed to explain it, and to show the manner of applying it to the purposes of computation.

The table, it will be observed, commences with 3,650 alive at age 20, which number is reduced by an annual decrement, which at each age up to 80 is equal to that age. The decrements consequently increase by a unit in passing from one age to the next higher. From and after 80 they follow a different law: they now, in passing from one age to the next, decrease by 5, and the number of living is in this manner exhausted at 96, which is thus the limiting age in the table. The numbers living thus form two consecutive series, having constant second differences—that of the earlier series being 1, and that of the later 5. From this description of the table it follows, as we shall see more fully presently, that the number living at any age may be expressed in terms of the age by an algebraical function of the second degree. The result with which we are here chiefly concerned, of this uniformity and simplicity of relation, is, that it admits of the easy application of known analytical methods for the summations constantly required in the computation of life contingencies, and which, in the case of previous tables in which this uniformity of relation does not hold, can only be effected by the aid of more or less voluminous auxiliary tables. By Mr. Orchard's table the value of any benefit, in which the number of lives concerned is any whatsoever, can be assigned without the aid of any auxiliary tables.

(1823), accompanying a presentation copy of Baron Maseres' *Scriptores Optici*. It represents, nearly, the Swedish Table of Mortality. It is of course empirical:

$$6199.8 - 9.29 \frac{x}{1} - 1.5767 \frac{x(x-1)}{2},$$

* *Phil. Trans.* 1825.

† *Life Tables, founded upon the discovery of a Numerical Law, &c.*, 1832.

‡ *Fifth Report of the Registrar-General, Appendix*, pp. 342, &c.

I proceed to show the use of this table in its application to single lives—more space than is at present at command being requisite to show the same in regard to two or more lives.

I first seek an analytical representation of the table—that is, an expression for the number living in terms of the age.

Let l_x denote the tabular number alive at age x , and d_x the number who die out of these before the attainment of age $x+1$.

Hence, $l_{x+1} - l_x$, or $\Delta l_x = -d_x$; ∴ integrating, $l_x = C - \Sigma d_x$, where C is a constant, to be determined by our knowledge of the value of the integral in a particular case. Now, for ages 80 and upwards d_x is obviously equal to $5(96-x)$, the finite integral of which is

$$\begin{aligned} 5\Sigma(96-x) &= -\frac{5}{2}(96-x)(97-x). \\ \therefore l_x &= C + \frac{5}{2}(96-x)(97-x). \end{aligned}$$

To determine C .—We know that $l_{96}=0$: and as in this case the formula becomes

$$l_{96} = C, \therefore 0 = C;$$

and the complete integral is

$$l_x = \frac{5}{2}(96-x)(97-x).$$

If $x=80$, $l_{80} = \frac{5}{2} \cdot 16 \cdot 17 = 40 \cdot 17 = 680$. We shall have occasion for this last result presently.

For ages 80 and downwards $d_x=x$, the integral of which is

$$\begin{aligned} \frac{x(x-1)}{2}; \\ \therefore l_x &= C - \frac{x(x-1)}{2}. \end{aligned}$$

To determine C .—We know, as above, that $l_{80}=680$, while for this age the expression just deduced becomes

$$\begin{aligned} l_{80} &= C - 3160, \\ \therefore 680 &= C - 3160, \text{ and } C = 3840; \end{aligned}$$

consequently the complete integral here is

$$l_x = 3840 - \frac{x(x-1)}{2}.$$

These two expressions, therefore,

$$l_x = \frac{5}{2}(96-x)(97-x),$$

and

$$l_x = 3840 - \frac{1}{2}x(x-1),$$

represent the mortality table, the former being applicable from ages 96 to 80, and the latter from 81 to 20—in both cases inclusive. Both formulæ hold consequently for ages 80 and 81.*

It may be worth while to remark also, that since for $x=0$ we get from the first expression

$$l_0 = 23280,$$

and from the second,

$$l_0 = 3840,$$

these would be respectively the radices of the two tables that would be formed by continuing to age 0 (that is, to birth) the laws implied in the foregoing expressions for l_x . The former, however, it has been seen, does not hold below 80, and it is not proposed to apply the latter below age 20.

Having thus the means of expressing the number living at each age as a function of the age, there is of course no difficulty in expressing in like manner the probability of a life at each age surviving any number of years. But I pass from this to what is of more importance.

Let S_x denote the sum of the series $l_x + l_{x+1} + \dots + l_{95}$, and let S_{x+1} denote the same for $l_{x+1} + l_{x+2} + \dots + l_{95}$. Then since

$$\begin{aligned} S_{x+1} - S_x &= \Delta S_x = -l_x, \\ \therefore S_x &= C - \Sigma l_x. \end{aligned}$$

Since l_x is not the same function of x throughout the limits to which this integral has reference, the integral must, for ages below 80, be considered in two portions. Let first x be not less than 80.

$$\therefore \Sigma l_x = \Sigma \frac{5}{2}(96-x)(97-x)(98-x) = -\frac{5}{6}(96-x)(97-x)(98-x),$$

$$\therefore S_x = C + \frac{5}{6}(96-x)(97-x)(98-x).$$

And since this ought to vanish for $x=96$, we thence obtain

$$C=0.$$

Hence the complete integral is

$$(x=\text{or } > 80) \quad S_x = \frac{5}{6}(96-x)(97-x)(98-x).$$

If $x=80$, we get for the sum of $l_{80} + l_{81} + \dots + l_{95}$,

$$S_{80} = 4080.$$

Let now $x < 80$. In this case the integral must be taken

* It will be found that the quadratic formed by equating to each other the two values of l_x has for its roots 80 and 81.

between the limits x and 80, and the result added to the value of S_{80} obtained above. We thus have

$$S_x = -\Sigma_x^{80} l_x + S_{80} = -(\Sigma l_x - \Sigma l_{80}) + 4080.$$

$$\Sigma l_x = \Sigma [3840 - \frac{1}{2}x(x-1)] = 3840x - \frac{1}{6}x(x-1)(x-2);$$

and making in this expression $x=80$, we get

$$\begin{aligned}\Sigma l_{80} &= 225040; \\ \therefore S_x &= \frac{1}{6}x(x-1)(x-2) - 3840x + 229120.\end{aligned}$$

If now e_x denote the mean duration of life at age x , we shall have

$$e_x = \frac{l_x + l_{x+1} + \dots}{l_x} - \frac{1}{2} = \frac{S_x}{l_x} - \frac{1}{2}.$$

Hence, using for S_x the proper one of the two expressions just deduced, according as x is or is not greater than 79, the mean duration at any age may be determined.

The mean durations for all ages are given in the appended table, and having been computed by the usual method they will afford verifications for the formulæ just deduced.

In the determination of the values of benefits dependent on the contingencies of human life, in accordance with any table of mortality, the sums of such series as the following are in constant requisition :

$$b_1v + b_2v^2 + b_3v^3 + \dots$$

in which v is the present value of one pound due in a year, and $b_1, b_2, \&c.$, are successive values of some function of the tabular numbers of living or dying, or both. In the case of the mortality tables usually employed, all these summations have in effect to be previously performed before the tables can be applied to use ; and their results are presented in the form of values of annuities and assurances, or of columns designated by the letters D, C, &c. The necessity for this preliminary formation arises from the circumstance that the values in the tables in question are not connected by any definite and easily assignable law. Were they so connected, the resources of analysis are amply sufficient to enable us to assign at once the value of any benefit that might be proposed, and so to dispense with the preliminary labour to which reference has been made. Such a connection, as we have seen, subsists in the case of the constants of Mr. Orchard's proposed table, and I am now to show its application to the valuation of life contingencies.

The theorem by which the requisite summations are to be effected may be thus enunciated :—

THEOREM.

If u_x be any function of x , of the form

$$u_x = b_1x + b_2x^2 + b_3x^3 + \dots \text{ in inf.,}$$

then u_x may also be written in the form

$$u_x = \frac{b_1x}{1-x} + \frac{\Delta b_1 x^2}{(1-x)^2} + \frac{\Delta^2 b_1 x^3}{(1-x)^3} + \dots,$$

which, if b_n be a rational and integer function of n , will necessarily be finite, since in that case the differences of b_1 will ultimately vanish.

This theorem is but a particular case of a theorem of very much greater generality, which will be found demonstrated and exemplified on pp. 239, 240 of De Morgan's *Differential and Integral Calculus*. The connection will be immediately apparent when it is remembered that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots,$$

and that $\frac{1}{(1-x)^2}$, $\frac{2}{(1-x)^3}$, &c., are the successive differential co-

efficients of $\frac{1}{1-x}$. The case in hand admits, however, of a more

elementary demonstration than that required for the theorem in its greatest generality; but it does not seem necessary to occupy space by inserting it here.

It will prepare us for the application of this theorem to life contingencies if we apply it first to annuities certain.

If $b_1, b_2, b_3, \&c.$ denote the successive payments of an annuity, its present value, to infinity, will be

$$b_1v + b_2v^2 + b_3v^3 + \dots \text{ in inf.,}$$

which, by the theorem, is equal to

$$\frac{b_1v}{1-v} + \frac{\Delta b_1 v^2}{(1-v)^2} + \frac{\Delta^2 b_1 v^3}{(1-v)^3} + \dots,$$

which becomes, since $v \div (1-v) = 1+r$,

$$\frac{b_1}{r} + \frac{\Delta b_1}{r^2} + \frac{\Delta^2 b_1}{r^3} + \dots$$

The value to n terms will be determined by deducting from this expression the sum of the $(n+1)^{\text{th}}$ and following terms.

This sum is

$$b_{n+1}v^{n+1} + b_{n+2}v^{n+2} + \dots$$

or

$$v^n(b_{n+1}v + b_{n+2}v^2 + \dots),$$

which, applying the theorem, is equal to

$$v^n\left(\frac{b_{n+1}}{r} + \frac{\Delta b_{n+1}}{r^2} + \dots\right).$$

Hence the required value to n terms, that is, the value of the annuity for n years, is

$$\frac{b_1}{r} + \frac{\Delta b_1}{r^2} + \dots - v^n\left(\frac{b_{n+1}}{r} + \frac{\Delta b_{n+1}}{r^2} + \dots\right);$$

and this is, beyond question, the most general annuity theorem that has yet been given.

Let first $b_1=b_2=b_3=\dots=1$; *i.e.*, let the annuity be a uniform annuity of one pound. Then are $b_1=1$, $b_{n+1}=1$, $\Delta b_1=0$, $\Delta^2 b_1=0$, &c. Hence the required value is

$$\frac{1}{r} - \frac{v^n}{r} = \frac{1-v^n}{r}.$$

Again, let the annuity be one whose successive payments are 1, 2, 3, &c. pounds; *i.e.*, whose n^{th} payment is n .

$$\begin{array}{ll} \therefore b_1=1 & b_{n+1}=n+1 \\ b_2=2 & 1 \\ & 0 \\ b_3=3 & b_{n+2}=n+2 \\ & 1 \\ & 0 \\ \therefore \Delta b_1=1, \Delta^2 b_1=0; & \Delta b_{n+1}=1, \Delta^2 b_{n+1}=0. \end{array}$$

Substituting, we have for the value required,

$$\frac{1}{r} + \frac{1}{r^2} - v^n\left(\frac{n+1}{r} + \frac{1}{r^2}\right),$$

or

$$\frac{1-(n+1)v^n}{r} + \frac{1-v^n}{r^2}.$$

This formula, by reduction, becomes

$$\frac{v+nv^{n+2}-(n+1)v^{n+1}}{(1-v)^2},$$

which is the expression usually given. Of the two, that which we have found above will, in almost all cases, be the more convenient for computation.

As a third example, let the annuity be one whose successive

payments are the squares of the natural numbers, commencing with unity; *i.e.*, whose n^{th} payment is n^2 .

$$\begin{array}{ll} \therefore b_1 = 1 & b_{n+1} = (n+1)^2 = n^2 + 2n + 1 \\ b_2 = 3 & b_{n+2} = (n+2)^2 = n^2 + 4n + 4 \\ b_3 = 4 & b_{n+3} = (n+3)^2 = n^2 + 6n + 9 \\ & \quad \quad \quad 2n+3 \quad 2 \\ & b_4 = 9 & b_{n+4} = (n+4)^2 = n^2 + 8n + 16 \\ & \quad \quad \quad 2 & \quad \quad \quad 2n+5 \quad 0 \\ & b_5 = 16 & \end{array}$$

$$\therefore \Delta b_1 = 3, \Delta^2 b_1 = 2, \Delta^3 b_1 = 0; \Delta b_{n+1} = 2n + 3, \Delta^2 b_{n+1} = 2, \Delta^3 b_{n+1} = 0.$$

Hence the value required is,

$$\frac{1}{r} + \frac{3}{r^2} + \frac{2}{r^3} - v^n \left(\frac{(n+1)^2}{r} + \frac{2n+3}{r^2} + \frac{2}{r^3} \right),$$

or

$$\frac{1-(n+1)v^n}{r} + \frac{3-(2n+3)v^n}{r^2} + \frac{2(1-v^n)}{r^3}.$$

And if, as a final example, the annuity whose payments are the cubes of the natural numbers (and consequently whose n^{th} payment is n^3) were proposed, we should find for its value, by a process analogous to the above,

$$\frac{1-(n+1)^3v^n}{r} + \frac{7-(3n^2+9n+7)v^n}{r^2} + \frac{12-6(n+2)v^n}{r^3} + \frac{6(1-v^n)}{r^4}.$$

For the purpose now in view it is necessary to investigate expressions for the sums of the series

$$d_x v + d_{x+1} v^2 + \dots + d_{95} v^{96-x}$$

and

$$l_x v + l_{x+1} v^2 + \dots + l_{95} v^{96-x};$$

and in consequence of the break in the mortality table we must, as before, consider each of them in two portions.

First, then, for the series $d_x v + d_{x+1} v^2 + \dots$; and let x be not less than 80. We have seen that for series involving v , the general theorem takes the form

$$\frac{b_1}{r} + \frac{\Delta b_1}{r^2} + \dots - v^n \left(\frac{b_{n+1}}{r} + \frac{\Delta b_{n+1}}{r^2} + \dots \right).$$

Hence, n being $96 - x$, we have for the sum required

$$\frac{d_x}{r} + \frac{\Delta d_x}{r^2} + \dots - v^{96-x} \left(\frac{d_{96}}{r} + \frac{\Delta d_{96}}{r^2} + \dots \right).$$

But

$\Delta d_x = -5$, $\Delta^2 d_x = 0$; also, $d_{96} = 0$, $\Delta d_{96} = -5$, and $\Delta^2 d_{96} = 0$.

The above becomes, therefore,

$$\frac{d_x}{r} - \frac{5}{r^2} + \frac{5v^{96-x}}{r^2} = \frac{d_x}{r} - \frac{5(1-v^{96-x})}{r^2}.$$

If $x=80$, this becomes

$$\frac{d_{80}}{r} - \frac{5(1-v^{16})}{r^2}, \text{ or } \frac{80}{r} - \frac{5(1-v^{16})}{r^2}.$$

Let now $x < 80$. We must sum this from x to 79, and add the result to the last when $x=80$, multiplied by v^{80-x} . The sum is

$$\frac{d_x}{r} + \frac{\Delta d_x}{r^2} + \dots - v^{80-x} \left(\frac{d_{80}}{r} + \frac{\Delta d_{80}}{r^2} + \dots \right).$$

Here

$$\Delta d_x = 1, \Delta^2 d_x = 0, d_{80} = 80, \Delta d_{80} = 1, \Delta^2 d_{80} = 0;$$

∴ the sum is

$$\frac{d_x}{r} + \frac{1}{r^2} - v^{80-x} \left(\frac{80}{r} + \frac{1}{r^2} \right);$$

to which adding, as above, the sum from 80 to 95, we get finally for the sum from x to 95,

$$\frac{d_x}{r} + \frac{1}{r^2} - v^{80-x} \left(\frac{80}{r} + \frac{1}{r^2} \right) + v^{80-x} \left(\frac{80}{r} - \frac{5(1-v^{16})}{r^2} \right),$$

or

$$\frac{d_x}{r} + \frac{1}{r^2} - \frac{[1+5(1-v^{16})]v^{80-x}}{r^2}.$$

The quantity $[1+5(1-v^{16})]/r^2$ is independent of the age, and if we denote it by Q , the expression just found will take the more convenient form

$$\frac{d_x}{r} + \frac{1}{r^2} - Qv^{80-x}.$$

Next, to assign the sum of

$$l_x v + l_{x+1} v^2 + \dots + l_{95} v^{96-x};$$

And, *first*, let x be not less than 80. Then by the theorem the sum is

$$\frac{l_x}{r} + \frac{\Delta l_x}{r^2} + \frac{\Delta^2 l_x}{r^3} + \dots - v^{96-x} \left(\frac{l_{96}}{r} + \frac{\Delta l_{96}}{r^2} + \frac{\Delta^2 l_{96}}{r^3} + \dots \right).$$

But

$$\Delta l_x = -d_x, \Delta^2 l_x = 5, \Delta^3 l_x = 0, l_{96} = 0, \Delta l_{96} = 0, \Delta^2 l_{96} = 5, \Delta^3 l_{96} = 0.$$

Hence the above becomes

$$\frac{l_x}{r} - \frac{d_x}{r^2} + \frac{5}{r^3} - \frac{5v^{96-x}}{r^3} = \frac{l_x}{r} - \frac{d_x}{r^2} + \frac{5(1-v^{96-x})}{r^3}.$$

If $x=80$ we hence have, for the sum of $l_{80}v + \dots + l_{95}v^{16}$,

$$\frac{680}{r} - \frac{80}{r^2} + \frac{5(1-v^{16})}{r^3}.$$

Let now $x < 80$. We must here, as before, sum from x to 79, and add the above sum, 80 to 95, multiplied by v^{80-x} . By the theorem the sum is

$$\frac{l_x}{r} + \frac{\Delta l_x}{r^2} + \frac{\Delta^2 l_x}{r^3} + \dots - v^{80-x} \left(\frac{l_{80}}{r} + \frac{\Delta l_{80}}{r^2} + \frac{\Delta^2 l_{80}}{r^3} + \dots \right).$$

But here $\Delta l_x = -d_x$, $\Delta^2 l_x = -1$, $\Delta^3 l_x = 0$, $l_{80} = 680$, $\Delta l_{80} = -d_{80} = -80$, $\Delta^2 l_{80} = -1$, $\Delta^3 l_{80} = 0$, &c. Hence the above becomes

$$\frac{l_x}{r} - \frac{d_x}{r^2} - \frac{1}{r^3} - \left(\frac{680}{r} - \frac{80}{r^2} - \frac{1}{r^3} \right) v^{80-x};$$

and adding to this the sum 80 to 95, as above, multiplied by v^{80-x} , we get finally for the sum required,

$$\frac{l_x}{r} - \frac{d_x}{r^2} - \frac{1}{r^3} - \left(\frac{680}{r} - \frac{80}{r^2} - \frac{1}{r^3} \right) v^{80-x} + \left(\frac{680}{r} - \frac{80}{r^2} + \frac{5(1-v^{16})}{r^3} \right) v^{80-x},$$

or

$$\frac{l_x}{r} - \frac{d_x}{r^2} - \frac{1}{r^3} + \frac{[1+5(1-v^{16})]v^{80-x}}{r^3};$$

which since $\frac{[1+5(1-v^{16})]}{r^2} = Q$, becomes

$$\frac{l_x}{r} - \frac{d_x}{r^2} - \frac{1}{r^3} + \frac{Q}{r} v^{80-x}.$$

Now, if we denote by A_x the present value of an assurance on (x) , and by a_x that of an annuity on (x) , we know that

$$A_x = \frac{d_x v + d_{x+1}v^2 + d_{x+2}v^3 + \dots}{l_x},$$

and

$$a_x = \frac{l_{x+1}v + l_{x+2}v^2 + l_{x+3}v^3 + \dots}{l_x};$$

hence, by what has just been established,

(x not less than 80)

$$A_x = \frac{1}{l_x} \left(\frac{d_x}{r} - \frac{5(1-v^{96-x})}{r^2} \right),$$

$$a_x = \frac{1}{l_x} \left(\frac{l_{x+1}}{r} - \frac{d_{x+1}}{r^2} + \frac{5(1-v^{96-x+1})}{r^3} \right);$$

$(x \angle 80)$

$$\Lambda_x = \frac{1}{l_x} \left(\frac{d_x}{r} + \frac{1}{r^2} - Qv^{80-x} \right)$$

$$a_x = \frac{1}{l_x} \left(\frac{l_{x+1}}{r} - \frac{d_{x+1}}{r^2} - \frac{1}{r^3} + \frac{Q}{r} v^{80-x+1} \right).$$

These expressions might be rendered more elementary, so as to be entirely independent of the mortality table, by substituting in them for l_x , d_x , &c., the values of these quantities in terms of x , the age; but this course does not on the whole seem advisable, at least as regards the numbers living. If for the numbers dying, however, we substitute their values in terms of x , the formulæ deduced become :

 $(x \text{ not less than } 80)$

$$\Lambda_x = \frac{5}{l_x} \left(\frac{96-x}{r} - \frac{1-v^{96-x}}{r^2} \right)$$

$$a_x = \frac{1}{l_x} \left\{ \frac{l_{x+1}}{r} - 5 \left(\frac{95-x}{r^2} - \frac{1-v^{95-x}}{r^3} \right) \right\}$$

 $(x \angle 80)$

$$\Lambda_x = \frac{1}{l_x} \left(\frac{x}{r} + \frac{1}{r^2} - Qv^{80-x} \right)$$

$$a_x = \frac{1}{l_x} \left(\frac{l_{x+1}}{r} - \frac{x+1}{r^2} - \frac{1}{r^3} + \frac{Q}{r} v^{79-x} \right).$$

To facilitate the practical application of these formulæ it will be well to be provided with the values of Q and $Q \div r$ for different rates of interest. The following table contains these values for the rates of 3, 4, 5 and 6 per cent.

	Q .	$Q \div r$.
3 per cent.	3204.6281	106820.94
4 "	2081.5369	52038.42
5 "	1483.7770	29675.54
6 "	1119.9357	18665.59

The following are examples of the application of the preceding formulæ.

Required the present value of an assurance and an annuity on (90), at 3 per cent.

$$\Lambda_{90} = \frac{1}{l_{90}} \left(\frac{d_{90}}{r} - \frac{5(1-v^6)}{r^2} \right).$$

$100r=3)3000=100d_{90}$	$\frac{1000}{902.8652}$	$\frac{83748426=v^6}{\cdot 16251574}$
$l_{90}=105)97.1348(\cdot 925093=A_{90}$	$\frac{263}{534}$	$\frac{9)8125.7870}{902.8652}$
	$\frac{98}{3}$	
		$a_{90} = \frac{1}{l_{90}} \left(\frac{l_{91}}{r} - \frac{d_{91}}{r^2} + \frac{5(1-v^5)}{r^3} \right)$
$100r=3)7500=100l_{91}$	$9)250000=\frac{10000d_{91}}{27777.78}$	$\frac{86260878=v^5}{\cdot 13739122}$
$\frac{2500}{25442.82}$	$\frac{27777.78}{27942.82}$	$\frac{5}{3)26328.46}$
$l_{90}=105)165.04(1.5718=a_{90}$	$\frac{600}{754}$	$\frac{686956.10}{25442.82}$
	$\frac{19}{8}$	

Required the present value of an assurance and an annuity on (50), at 3 per cent.

$A_{50} = \frac{1}{l_{50}} \left(\frac{d_{50}}{r} + \frac{1}{r^2} - Qv^{30} \right)$		
$3)5000=100d_{50}$	$9)10000$	$\frac{3.5057776=\log. Q}{\frac{1}{1.6148833}=,, v^{30}}$
$\frac{1666.667}{1111.111}$	$\frac{1111.111}{1320.265}$	$\frac{3.1206609}{}$
2777.778		
1320.265		
$2615)1457.513(\cdot 557366=A^{50}$		
15001		
19263		
958		
173		
16		
$a_{50} = \frac{1}{l_{50}} \left(\frac{l_{51}}{r} - \frac{d_{51}}{r^2} - \frac{1}{r^3} + \frac{Qv^{29}}{r} \right)$		
$3)256500=100l_{51}$	$9)510000=10000d_{51}$	$5.0286564=\log. \frac{Q}{r}$
$\frac{85500}{45329.08}$	$\frac{56666.67}{3)111111.11}$	$\frac{1.6277205=,, v^{29}}{\log. 45329.08 \}$
130829.08		
93703.71		
$2615)37125.37$	$\frac{37037.04=1 \div r^3}{56666.67}$	
$14.19708=a_{50}$	$\frac{56666.67}{93703.71}$	

The divisions here can of course be performed by logarithms, if preferred; or, still more readily, by means of the reciprocals of l_x , which will be found in Barlow.

The subjoined table contains, besides the mortality table, a number of deductions from it, involving, where interest is concerned, the four rates of 3, 4, 5 and 6* per cent. The several columns have been formed in the usual way, and consequently afford abundant means of testing the foregoing formulæ. In applying them for this purpose, however, it must be noted that, logarithms of six places only having been employed in the construction, there may be an occasional error of one, perhaps two, in the sixth figure of the tabulated values, where so many as six figures are given.

But the object mainly in view in the formation of the subjoined table was, to admit of a full comparison being made between the results of Mr. Orchard's hypothetical mortality table and those deduced from other tables founded on real observations; and so an opinion to be formed as to the suitableness of the table in question to be employed in the transaction of business. On such a comparison being instituted, it will be found that, amongst the tables chiefly had in estimation, the one which Mr. Orchard's most nearly resembles is the table known as Davies's Equitable. And the resemblance is very close indeed: in point of fact, I know that Mr. Orchard entertained a high opinion of the table referred to, and that he sought to assimilate his own to it, so far as the conditions to be fulfilled would permit. And as it is well known that Mr. Davies's table, in its main features, closely resembles the Carlisle Table—avoiding its anomalies,† however—it does not seem necessary here to add more on this point.

In what precedes I have endeavoured briefly to show the advantage attendant on the use of a table constituted as Mr. Orchard's—which is, that it admits, by the application of simple analytical processes, of the *independent* formation of any of the values which ordinarily require the aid of a formidable array of the results of previous computation. This advantage would be still more apparent did space permit the application of the principles here employed to the establishing of the formulæ for the various cases

* I am indebted to Mr. Charles Watkins, of the Pelican Life Office, for the formation of the column of annuities at 6 per cent.

† These are the irregularities arising from its want of graduation, and the extraordinary increase in the value of life during the ten years that follow age 87. Mr. Milne's remarks on this last-named point (p. 554) appear to me to be altogether at variance with the sound judgment which is his general characteristic.

involving two or three lives. What has been done will sufficiently indicate the route to be followed by any one disposed to follow out the subject.

In conclusion I may add, that Mr. Orchard's papers point to a different method—if not *two* different methods—from that which I have sought to elucidate of working out the results of his table; but I have not found the indications they contain sufficiently definite to enable me to seize the idea he appears to have had in view.

x.	l_x .	d_x .	e_x .	3 per cent.			4 per cent.	5 per cent.	6 per cent.	x.
				a_x .	A _x .	π_x .				
20	3650	20	41·544	21·7273	.338040	.014874	18·3958	15·8543	13·8731	20
21	3630	21	40·770	21·5023	.344590	.015314	18·2370	15·7387	13·7865	21
22	3609	22	40·004	21·2763	.351174	.015765	18·0768	15·6218	13·6987	22
23	3587	23	39·247	21·0491	.357795	.016227	17·9152	15·5035	13·6097	23
24	3564	24	38·497	20·8204	.364453	.016702	17·7521	15·3837	13·5193	24
25	3540	25	37·754	20·5905	.371153	.017191	17·5873	15·2626	13·4277	25
26	3515	26	37·019	20·3591	.377893	.017692	17·4209	15·1397	13·3346	26
27	3489	27	36·291	20·1260	.384678	.018209	17·2528	15·0151	13·2400	27
28	3462	28	35·571	19·8915	.391510	.018740	17·0828	14·8888	13·1442	28
29	3434	29	34·856	19·6553	.398390	.019288	16·9110	14·7608	13·0461	29
30	3405	30	34·149	19·4174	.405320	.019852	16·7372	14·6308	12·9467	30
31	3375	31	33·448	19·1776	.412301	.020434	16·5614	14·4989	12·8454	31
32	3344	32	32·754	18·9361	.419337	.021034	16·3835	14·3647	12·7424	32
33	3312	33	32·065	18·6926	.426427	.021654	16·2036	14·2287	12·6374	33
34	3279	34	31·383	18·4472	.433578	.022295	16·0213	14·0905	12·5305	34
35	3245	35	30·707	18·1996	.440786	.022958	15·8367	13·9501	12·4215	35
36	3210	36	30·036	17·9501	.448057	.023644	15·6498	13·8073	12·3104	36
37	3174	37	29·371	17·6982	.455390	.024355	15·4604	13·6621	12·1970	37
38	3137	38	28·711	17·4442	.462789	.025091	15·2684	13·5144	12·0813	38
39	3099	39	28·057	17·1879	.470256	.025856	15·0739	13·3641	11·9632	39
40	3060	40	27·409	16·9292	.477793	.026649	14·8767	13·2112	11·8426	40
41	3020	41	26·765	16·6680	.485398	.027473	14·6766	13·0555	11·7194	41
42	2979	42	26·126	16·4043	.493079	.028331	14·4737	12·8969	11·5935	42
43	2937	43	25·493	16·1380	.500833	.029223	14·2680	12·7354	11·4649	43
44	2894	44	24·864	15·8692	.508664	.030154	14·0592	12·5708	11·3334	44
45	2850	45	24·240	15·5976	.516576	.031124	13·8473	12·4031	11·1989	45
46	2805	46	23·621	15·3232	.524565	.032136	13·6322	12·2323	11·0613	46
47	2759	47	23·007	15·0461	.532639	.033194	13·4139	12·0580	10·9204	47
48	2712	48	22·397	14·7660	.540794	.034301	13·1922	11·8803	10·7763	48
49	2664	49	21·791	14·4830	.549036	.035460	12·9672	11·6991	10·6287	49
50	2615	50	21·190	14·1971	.557365	.036676	12·7385	11·5142	10·4775	50
51	2565	51	20·594	13·9080	.565786	.037952	12·5063	11·3255	10·3227	51
52	2514	52	20·001	13·6159	.574295	.039293	12·2704	11·1331	10·1640	52
53	2462	53	19·413	13·3206	.582896	.040703	12·0308	10·9366	10·0013	53
54	2409	54	18·829	13·0220	.591590	.042190	11·7873	10·7361	9·8347	54

TABLE—*continued.*

x.	l_x .	d_x .	e_x .	3 per cent.			4 per cent.	5 per cent.	6 per cent.	x.
				a_x .	A $_x$.	π_x .				
							a_x .	a_x .	a_x .	
55	2355	55	18·249	12·7203	.600380	.043759	11·5399	10·5314	9·6638	55
56	2300	56	17·674	12·4152	.609266	.045416	11·2884	10·3224	9·4886	56
57	2244	57	17·102	12·1067	.618251	.047170	11·0329	10·1090	9·3089	57
58	2187	58	16·535	11·7950	.627331	.049030	10·7733	9·8911	9·1246	58
59	2129	59	15·972	11·4798	.636510	.051003	10·5094	9·6686	8·9355	59
60	2070	60	15·413	11·1612	.645790	.053102	10·2414	9·4414	8·7417	60
61	2010	61	14·858	10·8392	.655169	.055339	9·9690	9·2094	8·5427	61
62	1949	62	14·308	10·5138	.664647	.057726	9·6922	8·9725	8·3387	62
63	1887	63	13·761	10·1850	.674223	.060279	9·4111	8·7306	8·1294	63
64	1824	64	13·219	9·8329	.683896	.063015	9·1256	8·4838	7·9148	64
65	1760	65	12·682	9·5175	.693662	.065953	8·8357	8·2319	7·6948	65
66	1695	66	12·149	9·1790	.703526	.069116	8·5415	7·9750	7·4692	66
67	1629	67	11·621	8·8374	.713472	.072526	8·2431	7·7130	7·2382	67
68	1562	68	11·098	8·4930	.723505	.076215	7·9406	7·4461	7·0016	68
69	1494	69	10·580	8·1459	.733613	.080212	7·6341	7·1742	6·7595	69
70	1425	70	10·068	7·7966	.743790	.084555	7·3239	6·8977	6·5120	70
71	1355	71	9·563	7·4453	.754020	.089282	7·0103	6·6167	6·2593	71
72	1284	72	9·064	7·0927	.764290	.094442	6·6939	6·3317	6·0018	72
73	1212	73	8·573	6·7395	.774576	.100081	6·3752	6·0492	5·7398	73
74	1139	74	8·090	6·3866	.784858	.106255	6·0551	5·7520	5·4741	74
75	1065	75	7·617	6·0352	.795090	.113015	5·7349	5·4593	5·2058	75
76	990	76	7·157	5·6872	.805226	.120412	5·4162	5·1665	4·9362	76
77	914	77	6·710	5·3449	.815196	.128479	5·1012	4·8759	4·6674	77
78	837	78	6·281	5·0117	.824902	.137215	4·7933	4·5907	4·4026	78
79	759	79	5·876	4·6926	.834196	.146541	4·4973	4·3156	4·1463	79
80	680	80	5·500	4·3949	.842867	.156234	4·2206	4·0578	3·9057	80
81	600	75	5·167	4·1303	.850572	.165794	3·9747	3·8288	3·6921	81
82	525	70	4·833	3·8620	.858390	.176552	3·7242	3·5946	3·4727	82
83	455	65	4·500	3·5898	.866320	.188749	3·4690	3·3549	3·2474	83
84	390	60	4·167	3·3137	.874358	.202692	3·2091	3·1098	3·0161	84
85	330	55	3·833	3·0337	.882510	.218783	2·9441	2·8590	2·7781	85
86	275	50	3·500	2·7497	.890786	.237564	2·6742	2·6023	2·5338	86
87	225	45	3·167	2·4615	.899178	.259763	2·3992	2·3397	2·2827	87
88	180	40	2·833	2·1692	.907692	.286409	2·1190	2·0708	2·0245	88
89	140	35	2·500	1·8727	.916330	.318982	1·8334	1·7956	1·7592	89
90	105	30	2·167	1·5718	.925092	.359705	1·5423	1·5138	1·4863	90
91	75	25	1·833	1·2666	.933982	.412073	1·2456	1·2253	1·2057	91
92	50	20	1·500	·9568	.943008	.481908	·9432	·9299	·9170	92
93	30	15	1·167	·6425	.952160	.579689	·6349	·6274	·6200	93
94	15	10	·833	·3236	.961448	.726375	·3205	·3175	·3145	94
95	5	5	·500	·970874	.970874					95
96	0									96